Stationary solutions of non-autonomous symmetries of integrable equations

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Outline

Part I. The KdV equation

$$u_t = u_{xxx} + 6uu_x$$

- Motivation: Gurevich–Pitaevskii problems
- Stationary solutions for non-autonomous symmetries
- Higher symmetry + Galilean symmetry → Suleimanov ODE (4-th order)
- Master-symmetry + scaling → some 6-th order ODE
- Step-like solutions for very special initial data

Part II. Non-Abelian Volterra lattice equations

$$\mathrm{VL}^1 \quad u_{n,x} = u_{n+1}u_n - u_n u_{n-1}$$

$$\mathrm{VL}^2 \quad u_{n,x} = u^{\mathrm{T}}_{n+1}u_n - u_n u^{\mathrm{T}}_{n-1}$$

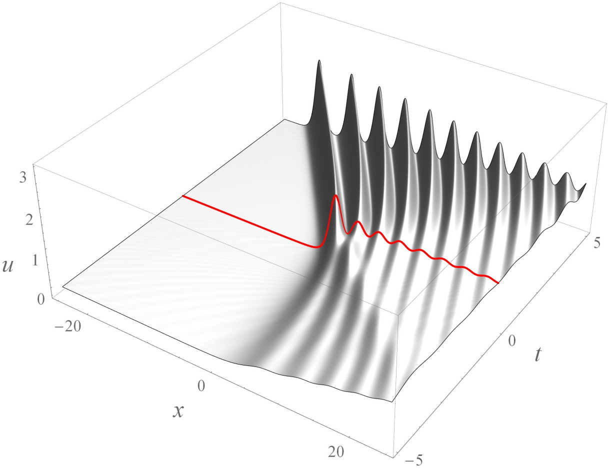
- Substitutions $\mathrm{VL}^1 \leftarrow \cdots \rightarrow \mathrm{VL}^2$
- Higher symmetry + scaling → $\mathrm{dP}^i_1 + \mathrm{P}^i_4$
- Master-symmetry + scaling + D_x → $\mathrm{dP}^i_{34} + \mathrm{P}^i_5$ $\quad i = 1, 2$
- Master-symmetry + D_x → $\mathrm{dP}^i_{34} + \mathrm{P}^i_3$

Part I

KdV equation

Main result (a conjecture)

The KdV equation $u_t = u_{xxx} + 6uu_x$ admits step-like solutions which satisfy certain nonautonomous ODE of sixth order (*J. Nonl. Math. Phys.* 2020).

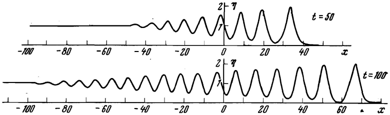


Motivation: Gurevich–Pitaevskii problems (1973)

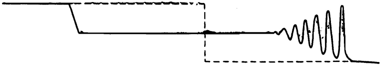
decay of the initial discontinuity

$$u(x,0)=\begin{cases}0, & x<0,\\ 1, & x\geq 0.\end{cases}$$

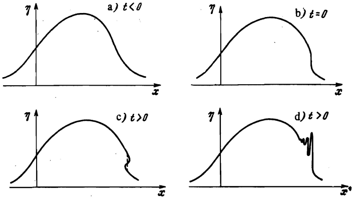
compression wave

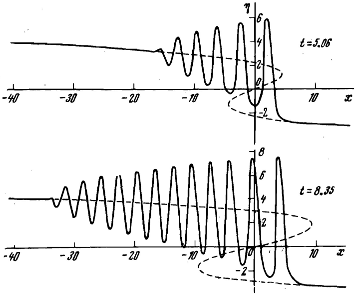


rarefaction and compression waves (in KdV, appear separately at $t\to\pm\infty$)



breaking of the wave front



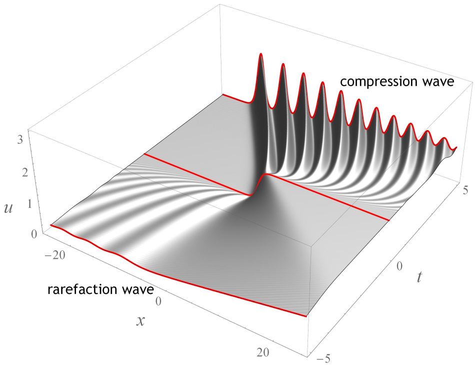


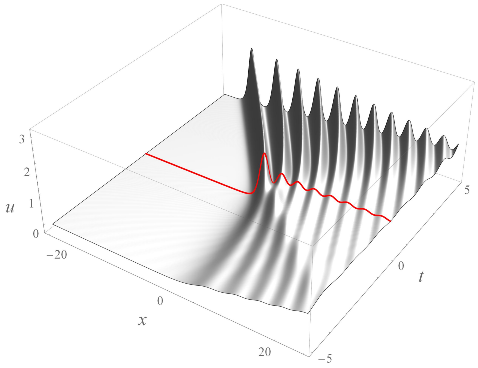
Step-like solutions

We are interested in the first GP problem for the KdV case.

- Notice that left and right steps are related by $(x,t) \to (-x,-t)$.
- A numerical solution can be obtained for rather generic initial data, for instance by use of the Zabuski–Kruskal scheme. Of course, such a solution must not satisfy any ODE.

An example for $u(x,0) = \frac{1}{2}(1 + \tanh x)$ compared with our solution:





Some known results

- Hruslov & Kotlyarov (1976, 1994), Venakides (1986): Inverse Scattering Method, asymptotic expansions
- Cohen (1984), Kappeler (1986) and others: study of correctness of the Cauchy problems with step-like and even more general initial data
- Bikbaev (1989), Novokshenov (2005), Egorova & Teschl (2013) and others: generalizations for initial data with finite-gap asymptotic and for other models

It is well-known that KdV admits a family of Galilean-invariant solutions described by P_1 and a family of scaling-invariant solutions described by P_2.

Solutions of both GP problems also exhibit a kind of self-similarity.

Although there are no any explicit Ansatz for these problems, it is natural to conjecture that some special solutions may be related with higher KdV symmetries.

Second GP problem (formation of the oscillating zone in the vicinity of the breaking point): this idea turned out to be correct and fruitful.

- Suleimanov & Kudashev (1994, 1996): a solution with required behaviour can be found among solutions of the stationary equation for the sum of higher symmetry of 5-th order and the Galilean symmetry

$$u_{t_5} + ku_{\tau_1} = 0 \quad \Leftrightarrow \quad u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3 + k(6tu + x) = 0$$

- Dubrovin (2006): the conjecture on the uniqueness of this solution
- Claeys, Vanlessen (2007): the existence proof

First GP problem (evolution of step-like initial data): our goal is to demonstrate, numerically, that it also admits solutions related with a sum of master-symmetry and the scaling symmetry.

In contrast to the above example, there is a 2-parametric family of such solutions.

Symmetries of the KdV equation

The commutative and non-commutative parts of the KdV hierarchy are

$$u_{t_{2j+1}} = R^j(u_x), \quad u_{\tau_{2j+1}} = R^j(6tu_x + 1), \quad j = 0, 1, 2, \dots$$

where $R = D_x^2 + 4u + 2u_x D_x^{-1}$ is the recursion operator.

Symmetry algebra ($\partial_{t_{2j+1}} \sim \lambda^j$, $\partial_{\tau_{2j+1}} \sim \lambda^j \partial_\lambda$):

$$[\partial_{t_{2j+1}}, \partial_{t_{2k+1}}] = 0, \qquad [\partial_{\tau_{2j+1}}, \partial_{t_{2k+1}}] = k\partial_{t_{2j+2k-1}},$$

$$[\partial_{\tau_{2j+1}}, \partial_{\tau_{2k+1}}] = (k - j)\partial_{\tau_{2j+2k-1}}.$$

- Ibragimov & Shabat (1979): ∂_{τ_5} flow (master-symmetry)
- Fuchssteiner (1983): the general concept of master-symmetry
- Orlov & Shulman (1985): additional symmetry algebra for NLS
- Burtsev, Zakharov & Mikhailov (1987): zero curvature representations with variable spectral parameter

All symmetries of order ≤ 5

This is all we need:

$$
\begin{aligned}
u_{t_1} &= u_x && (x\text{-translation})\\
u_{t_3} &= (u_{xx} + 3u^2)_x && (t\text{-translation})\\
u_{t_5} &= (u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3)_x && (\text{higher symmetry})\\
u_{\tau_1} &= 6tu_x + 1 && (\text{Galilean transform})\\
u_{\tau_3} &= 3tu_{t_3} + xu_x + 2u && (\text{scaling})\\
u_{\tau_5} &= 3tu_{t_5} + xu_{t_3} + 4u_{xx} + 8u^2 + 2u_x D_x^{-1}(u) && (\text{master-symmetry})
\end{aligned}
$$

Here, the nonlocal variable $v = D_x^{-1}(u)$ satisfies equations

$$v_x = u \qquad \text{and} \qquad v_t = u_{xx} + 3u^2.$$

Stationary equations

The stationary equation $E[u]=0$ for any symmetry satisfies the identity

$$D_t(E)=(D_x^3+6uD_x+6u_x)(E)=0,$$

that is, it defines a constraint consistent with KdV.

- Novikov, Dubrovin, Matveev (1974, 1976): finite-gap solutions, if we use only autonomous symmetries
- adding of non-autonomous symmetries leads to the Painlevé equations of their higher analogues
- Moore (1990) and others: string equations

The general form of stationary equation of order ≤ 5 is

$$k_0 u_{\tau_5} + k_1 u_{\tau_3} + k_2 u_{\tau_1} + k_3 u_{t_5} + k_4 u_{t_3} + k_5 u_{t_1} = 0.$$

- In fact, there are no essential parameters here, since all constants can be changed by point transformations. We only should distinguish between several cases.
- In particular, the case $k_0 = 0$ and $k_3 \neq 0$ leads to $u_{t_5} + k u_{\tau_1} = 0$ where either $k = 1$ (the Suleimanov equation) or $k = 0$ (2-gap solution).
- We are interested in the case $k_0 = 1$. First, we set $k_3 = k_4 = k_5 = 0$ by adding constants to t, x and v.

Proposition 1

PDE system

$$u_t = u_{xxx} + 6uu_x, \quad v_t = u_{xx} + 3u^2$$

is consistent with the ODE system

$$3t(u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3)_x + x(u_{xx} + 3u^2)_x + 4u_{xx} + 8u^2 + 2u_x v$$
$$+ k_1(3t(u_{xx} + 3u^2)_x + xu_x + 2u) + k_2(6tu_x + 1) = 0, \qquad v_x = u.$$

Notice that all constant solutions $u = c$ of this system are given by equation

$$8c^2 + 2k_1 c + k_2 = 0.$$

Its zeroes can be changed by the scaling and the Galilean transformations.

Two possibilities, which we do not consider:

- $c^2 + 1 = 0$ (no real solutions with constant asymptotic at all),
- $c^2 = 0$ (steps are not possible).

Since we wish to obtain solutions with different asymptotics 0 and 1 for $x \to \pm\infty$, we should take

- $c(c-1) = 0$ (this may, *potentially*, lead to a step).

Or, maybe not, but the only choice which we have to analyze is $k_1 = -4$ and $k_2 = 0$:

$$u_{\tau_5} - 4u_{\tau_3} = 0.$$

Isomonodromic Lax pairs and first integrals

We use compatibility conditions for linear equations

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi, \quad \Psi_\tau + \kappa(\lambda)\Psi_\lambda = W\Psi,$$

where $\kappa(\lambda)$ is a polynomial with constant coefficients. Equation

$$U_t = V_x + [V, U]$$

with

$$U = \begin{pmatrix} 0 & 1 \\ -\lambda - u & 0 \end{pmatrix}, \quad V = \begin{pmatrix} -u_1 & -4\lambda + 2u \\ 2(\lambda + u)(2\lambda - u) - u_2 & u_1 \end{pmatrix}$$

is equivalent to KdV. Any symmetry corresponds to a matrix W of the form

$$W[Y] = \begin{pmatrix} -Y_x & 2Y \\ -2(\lambda + u)Y - Y_{xx} & Y_x \end{pmatrix},$$

in particular, $U = W[1/2]$ and $V = W[-2\lambda + u]$.

The stationary equation $u_\tau = 0$ gives a pair of consistent ODE systems with (isomonodromic) Lax pairs

$$\kappa U_\lambda = W_x + [W, U], \quad \kappa V_\lambda = W_t + [W, V] \quad \Leftrightarrow$$

$$Y_{xxx} + 4(u + \lambda)Y_x + 2u_1 Y = \kappa, \qquad Y_t = Y_{xxx} + 6uY_x - 3\kappa.$$

- If $\kappa(\lambda) = 0$ (autonomous symetries) then the system is Liouville integrable since there is the first integral polynomial in λ:

$$H(\lambda) = \det W = 2YY_{xx} - Y_x^2 + 4(\lambda + u)Y^2 = \mathrm{const}(\lambda).$$

- If $\kappa(\lambda) \neq 0$ then the system is not integrable. There are only $\deg_\lambda \kappa$ first integrals corresponding to zeroes of κ:

$$H_i = H(\lambda_i), \quad \kappa(\lambda_i) = 0$$

(for a multiple zero, if $\kappa^{(j)}(\lambda_i) = 0$ for $j < r_i$ then $d^j H/d\lambda^j(\lambda_i)$ are also first integrals).

Our case is $u_{\tau_5} - 4u_{\tau_3} = 0$. This corresponds to

$$\kappa = -8\lambda(\lambda+1), \qquad Y = 24t\lambda^2 - 2(6tu + x - 12t)\lambda + y,$$
$$y = Y(0) = 3t(u_2 + 3u^2) + xu + u_{-1} - 2(6tu + x).$$

It is convenient to rewrite equations in terms of u and y.

Proposition 2

KdV equation admits solutions defined by the following pair of consistent ODE systems of sixth order:

$$\begin{cases} 3t(u_{xxx} + 6uu_x - 4u_x) + xu_x + 2u - y_x - 2 = 0, \\ y_{xxx} + 4uy_x + 2u_xy = 0, \end{cases} \tag{1}$$

$$\begin{cases} u_t = u_{xxx} + 6uu_x, \\ y_t = 2uy_x - 2u_xy. \end{cases} \tag{2}$$

These systems admit two first integrals in common:

$$H_0 = H(0) = 2yy_{xx} - y_x^2 + 4uy^2, \quad H_1 = H(-1) = 2zz_{xx} - z_x^2 + 4(u-1)z^2,$$

where $z = Y(-1) = 12tu + 2x + y$.

Few explicit solutions

u	y	H_0	H_1
0	$-2x+a$	-4	$-4a^2$
1	a	$4a^2$	-4
$\dfrac{2}{\cosh^2 X}$	$2(1-x\tanh X)\tanh X,\ \ X=x+4t+a$	-4	-16
$1-\dfrac{2}{\cos^2 X}$	$\dfrac{2(6t-x)-\sin 2X}{\cos^2 X},\ \ X=x+2t+a$	-16	-4
$-\dfrac{x}{6t}$	0	0	0
$-\dfrac{2}{(x+a)^2}$	$\dfrac{2(12t+a)}{(x+a)^2}-2(x+a)$	-36	$-16a^2$
$1-\dfrac{2}{(x+6t+a)^2}$	$\dfrac{2a}{(x+6t+a)^2}+2a$	$16a^2$	-36

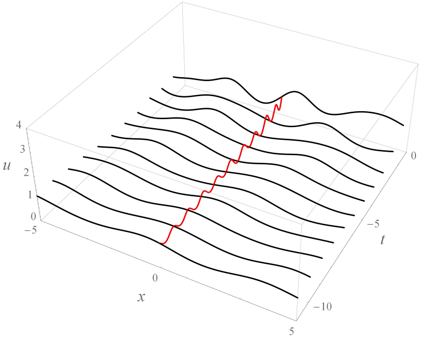
- The first three solutions are regular for all x, t. It is likely that there are no other regular solutions in closed form.
- There are infinitely many rational solutions which can be obtained by Bäcklund transformations.
- There are also some families of solutions in terms of the Bessel functions.
- Unfortunately, there are no explicit step-like solutions.

How to solve numerically?

To construct a solution in the half-plane $t<0$ (or $t>0$):

$$\text{Start from initial data } (u_0,u_1,u_2,y_0,y_1,y_2)\in\mathbb{R}^6 \text{ at } (x_0,t_0),\ t_0<0$$
$$\downarrow$$
$$\text{Solve (2) with respect to } t \text{ at } x=x_0. \text{ This gives initial data for (1).}$$
$$\downarrow$$
$$\text{Solve (1) with respect to } x \text{ for all } t\in(-\infty,0).$$

Vice versa, one can first solve (1) at $t=t_0$ and then use the solution as initial data for (2) for all x. The results coincide, if we do not meet a singularity. Experiments demonstrate that, for some domain in the space of initial data, solutions are regular for $x\in\mathbb{R}$ and $t\in\mathbb{R}_{<0}$.



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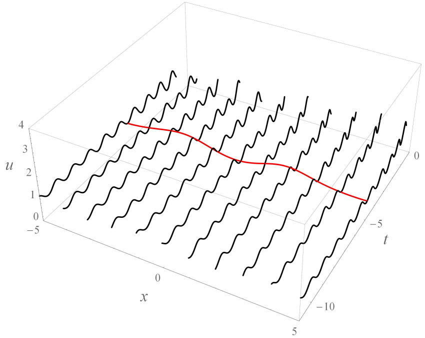

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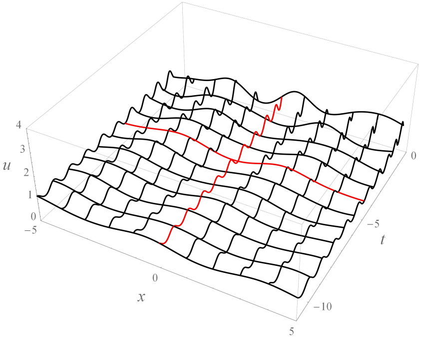


How to solve numerically?

To construct a solution in the half-plane $t < 0$ (or $t > 0$):

Start from initial data $(u_0, u_1, u_2, y_0, y_1, y_2) \in \mathbb{R}^6$ at (x_0, t_0), $t_0 < 0$

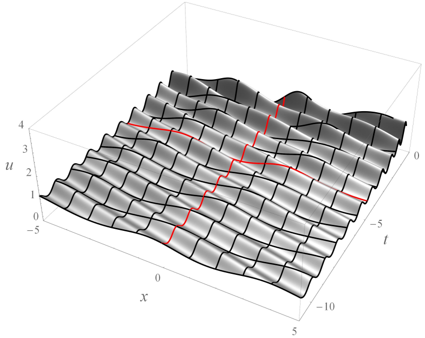
$$\downarrow$$

Solve (2) with respect to t at $x = x_0$. This gives initial data for (1).

$$\downarrow$$

Solve (1) with respect to x for all $t \in (-\infty, 0)$.

Vice versa, one can first solve (1) at $t = t_0$ and then use the solution as initial data for (2) for all x. The results coincide, if we do not meet a singularity. Experiments demonstrate that, for some domain in the space of initial data, solutions are regular for $x \in \mathbb{R}$ and $t \in \mathbb{R}_{<0}$.



Solutions for generic initial data are not very interesting. Such a solution does not have the desired asymptotics, moreover, it becomes singular along the line $t = 0$.

In order to obtain step-like solutions we have to go through a sieve of subsequent specializations.

generic solutions (6 parameters)

$\downarrow$

regularity condition for $t = 0$ (4 parameters)
the degenerate Painlevé equation P_5

$\downarrow$

regularity condition for $x = 0, t = 0$ (3 parameters)
explicit initial data

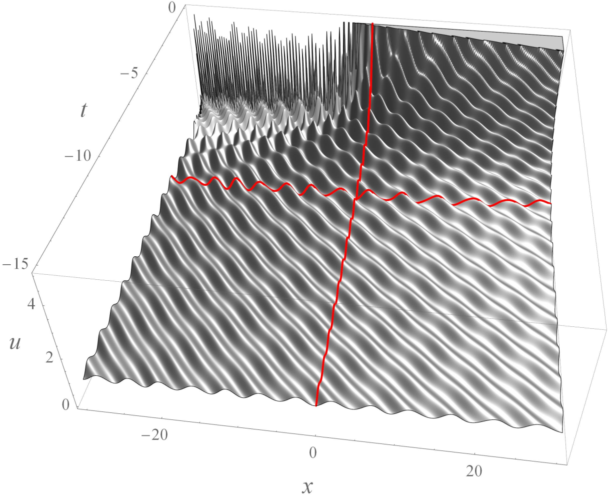
$\downarrow$

separatrix step-like solutions (2 parameters)
implicit special initial data by shooting method

Each stage is not very effective, because these special solutions are not stable and errors quickly lead to less degenerate ones.

Generic solutions

Because of the coefficient t at u_{xxx} in (1), the line $t=0$ is singular. A generic solution blows up along this line (a simplest explicit example is $u=-\dfrac{x}{6t}$).



Regularity condition at $t = 0$

In order to construct solutions which are regular at $t = 0$, we have to choose special initial data on this line which are subjected to a fourth order ODE:

$$\begin{cases} \sout{3t(u_{xxx} + 6uu_x - 4u_x)} + xu_x + 2u - y_x - 2 = 0, \\ y_{xxx} + 4uy_x + 2u_xy = 0. \end{cases} \tag{3}$$

The first integrals become

$$H_0 = 2yy_{xx} - y_x^2 + 4uy^2,$$
$$H_1 = 2(2x + y)y_{xx} - (2 + y_x)^2 + 4(u - 1)(2x + y)^2.$$

It is possible to eliminate u and to obtain a second order ODE for y. It is equivalent to degenerate P_5 with the coefficient $\delta = 0$.

Proposition 3 (V.A., Shabat, Yamilov 2000)

The system (3) is equivalent to P_5

$$p'' = \left(\frac{1}{2p} + \frac{1}{p-1}\right)(p')^2 - \frac{p'}{X} + \frac{(p-1)^2}{X^2}\left(\alpha p + \frac{\beta}{p}\right) + \gamma\frac{p}{X} + \delta\frac{p(p+1)}{p-1} \quad (\mathrm{P}_5)$$

with parameters

$$\alpha = -\frac{H_0}{32}, \quad \beta = \frac{H_1}{32}, \quad \gamma = \frac{1}{2}, \quad \delta = 0$$

under the change

$$p(X) = \frac{2x}{y(x)} + 1, \quad X = x^2.$$

In fact, equation P_5 with $\delta = 0$ is also related with a special case of P_3 (Gromak 1975).

This is nice and gives a hope that some advanced methods can be applied.

However, for now we just solve equations numerically in the original form (3).

Regularity condition at $x = 0$

In turn, (3) has the fixed singular point $x = 0$:

$$xu_x + 2u - y_x - 2 = 0, \quad y_{xxx} + 4uy_x + 2u_x y = 0.$$

In order to construct regular solutions we have to impose a constraint on the initial data at the origin:

$$2u(0,0) = y_x(0,0) + 2.$$

Effectively, the order of equations becomes equal to 3.
In a neighbourhood of $x = 0$, a solution is given by power series

$$y = a_0 + a_1 x + a_2 x^2 + \dots, \quad u = b_0 + b_1 x + b_2 x^2 + \dots,$$

where a_0, a_1 and a_2 are arbitrary and

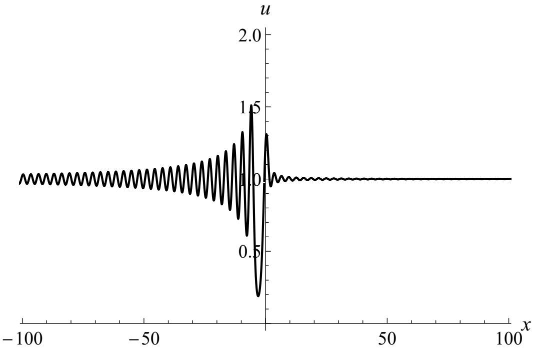
$$b_0 = 1 + \frac{1}{2}a_1, \quad b_{n-1} = \frac{n}{n+1}a_n, \quad n = 2, 3, \dots,$$

$$a_n = -\frac{2}{n(n-1)(n-2)} \sum_{j=0}^{n-2} (2n-4-j) b_j a_{n-2-j}, \quad n = 3, 4, \dots .$$

It turns out that the radius of convergence is not zero.

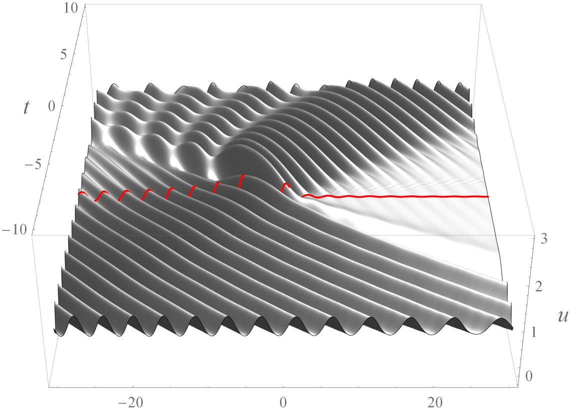
A typical regular solution (at $t = 0$)

- We use the series in some interval near the origin, then continue by Runge–Kutta method.
- Movable poles for $x \neq 0$ are possible, but there exists a domain in the space of initial data corresponding to regular solutions which are stable in both directions.
- A typical profile has the form of slowly decaying (like x^{-1}) oscillations near $u = 1$, separated by a well near the origin, with different oscillation amplitudes on the left and right.

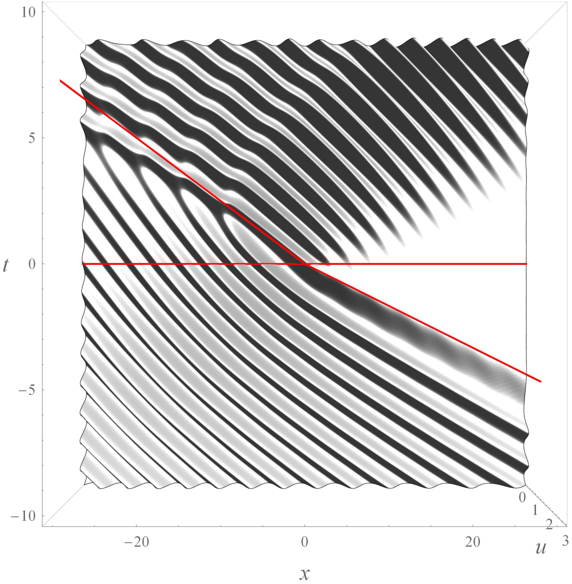


A typical regular solution (for all t)

Now we have to solve the t-part (2) for the obtained initial data. This stage is most difficult. Theoretically, this can be done in the same fashion, by constructing series in a neigbourhood of $t = 0$ and then solving numerically. It works, but not very well. In practice, the finite difference methods for PDE turns out to be more efficient. Anyway, here is a regular solution (with initial data shown on the previous figure):

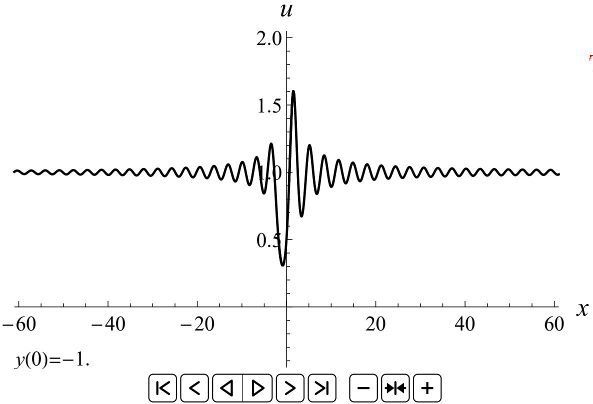


The same solution from above. The half-lines are $x = -6t$, $t < 0$ and $x = -4t$, $t > 0$.



But where is the step?

Let us vary the initial data for the system (3), very smoothly...........



Warning!
This is not the evolution in t

We fix the first integrals (here $H_0 = -2$ and $H_1 = -6$, for instance).

The only free parameter is a_0. Roughly speaking, to expand the well by 1, we need to calculate the next exact decimal digit of a_0.

Shooting method

The goal is to choose a_0 for fixed first integrals

$$H_0 = 4a_0a_2 + 2a_0^2(a_1+2) - a_1^2, \quad H_1 = 4a_0a_2 + 2a_0^2a_1 - (a_1+2)^2$$

(fixing a_1 and a_2 is practically the same).

- Start from an interval $[a_0(1), a_0(2)]$, such that the solution for one endpoint has a pole and the solution for another endpoint is oscillating.
- Remove one or another half of the interval, depending on the type of the solution in the middle.
- This yields a sequence

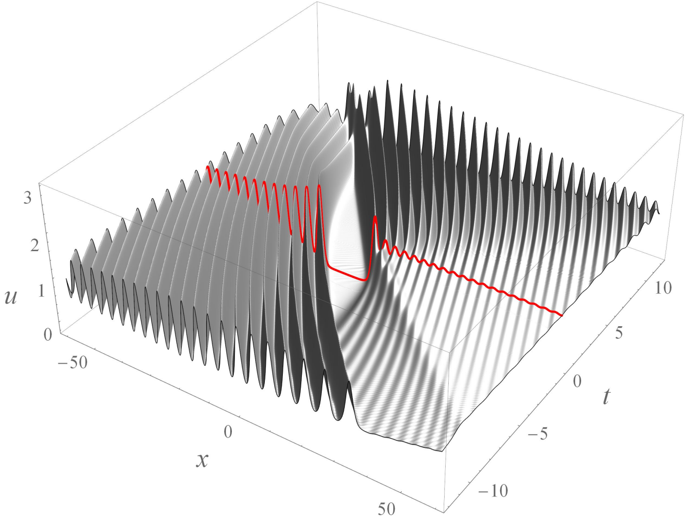
  $$a_0(n) \to a_0, \quad n = 1, 2, 3, \dots$$

  for which the well is gradually widening.

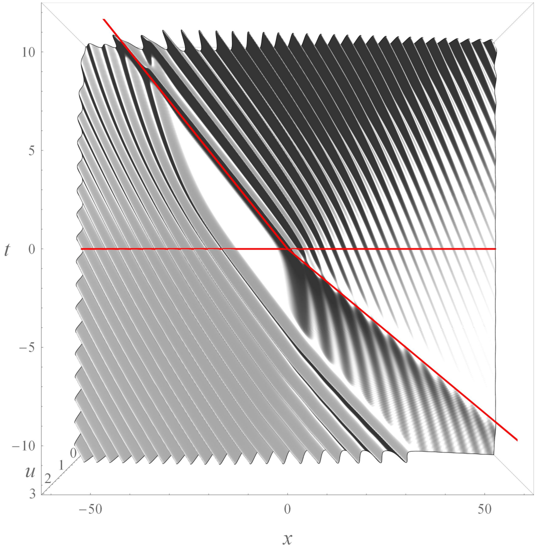
Solution remains practically unchanged on one half-line, but changes drastically on the another one: the oscillating zone in it moves farther and farther away from the origin.

This is slightly reminiscent of the limiting transition from the cnoidal wave to a soliton. The difference is that it pushes apart just two peaks, not all.

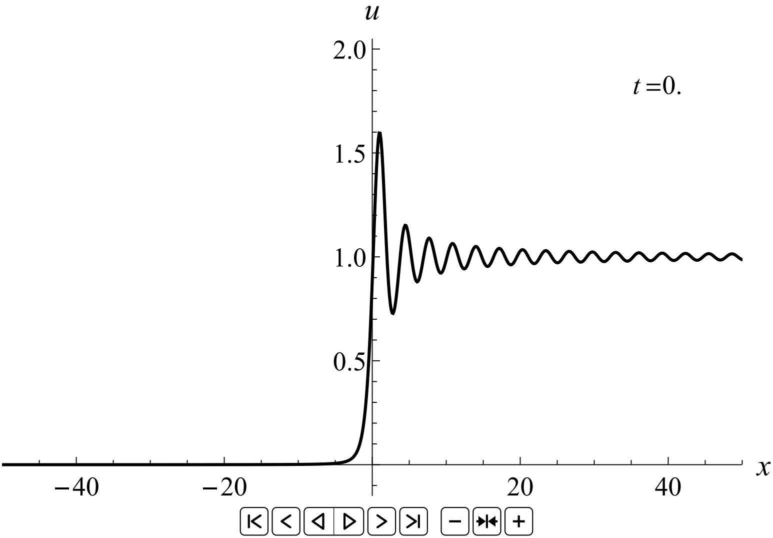
An intermediate solution with wide well



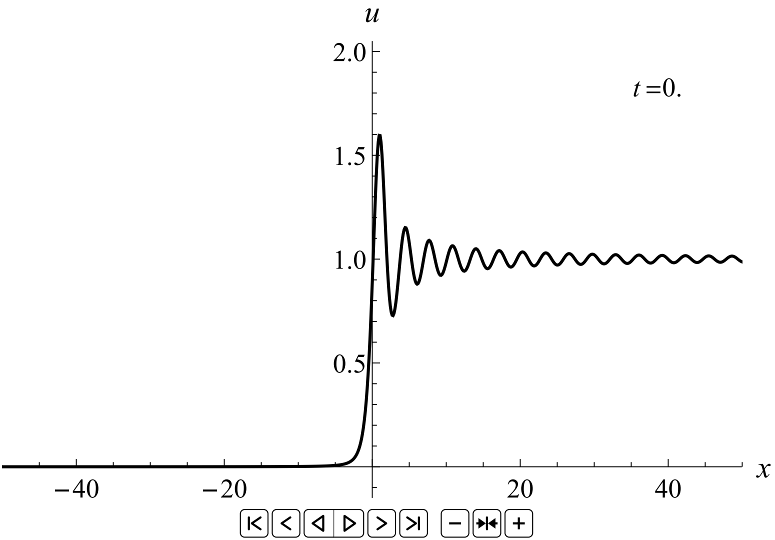
Viem from above

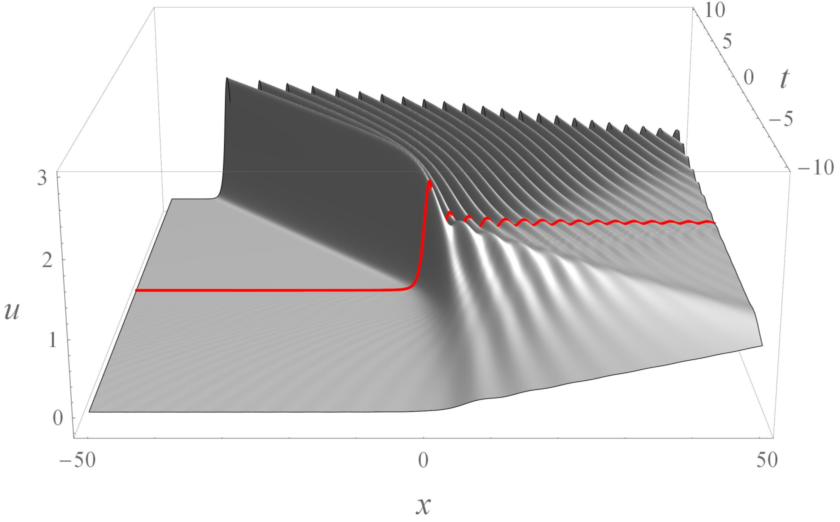


Step-like solution. The compression wave



The rarefaction wave




10
5
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t
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3
2
u
1
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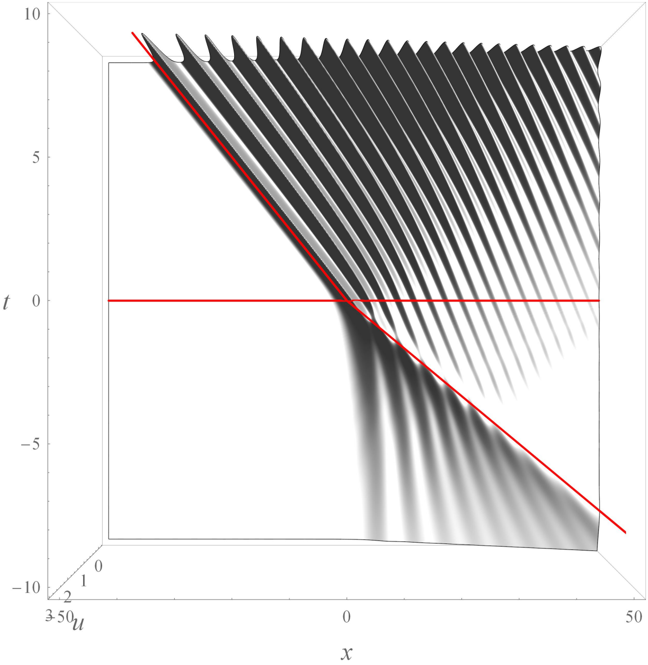
−50
0
50
x


10
5
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x

A conjecture

The steps corresponding to different values of H_0 and H_1 are very similar; the difference can be seen only in the asymptotic coefficients. For $u \to 0$ we have

$$u(x) \sim A_2 x^{-2} + A_3 x^{-3} + \dots, \quad y(x) \sim -2x + B_0 + B_2 x^{-2} + B_3 x^{-3} + \dots$$

It turns out that

$$A_2 = \frac{H_0 + 4}{16}, \quad B_0^2 = -\frac{H_1}{4},$$

and all other coefficients are uniquely defined. So, this asymptotic is defined by the first integrals.

Conjecture

Step-like solutions of (1), (2) with the asymptotic $u \to 0$, $x \to -\infty$ exist for some two values of a_0 for any $H_0 > -4$ and $H_1 < 0$.

This means existence of 2-parametric family of step-like solutions. The symmetric steps with $u(x) \to 0$, $x \to +\infty$ correspond to the values $-a_0$.

Part II

Non-Abelian Volterra lattices

Results

We study two integrable versions of non-Abelian Volterra lattice:

VL^1	$u_{n,x} = u_{n+1}u_n - u_nu_{n-1}$	(Salle 1982)
VL^2	$u_{n,x} = u^{\mathrm{T}}_{n+1}u_n - u_nu^{\mathrm{T}}_{n-1}$	(new? arXiv:2010.09021)

One can think of u_n as square matrices of arbitrary size or elements of some abstract algebra. Sometimes we will need inversion.

- For each of these equations, we derive constraints as stationary equations for simplest non-autonomous symmetries, including the master-symmetries.
- The result is some set of non-Abelian analogs of discrete and continuous Painlevé equations.

In the scalar case, these constraints were studied in (V.A. & Shabat 2019), where also some numerical results were presented.

In the non-Abelian case, no solutions for now, only equations.

$\mathrm{VL}^1 \leftarrow \mathrm{mVL}^1 \leftarrow \text{pot-mVL} \rightarrow \mathrm{mVL}^2 \rightarrow \mathrm{mVL}^2$

VL^1 and VL^2 are related, but not in an obvious way.

$$\begin{aligned}
\mathrm{VL}^1: &\quad u_{n,x} = u_{n+1}u_n - u_n u_{n-1} \\
\mathrm{mVL}^1: &\quad v_{n,x} = v_{n+1}(v_n^2 - \alpha^2) - (v_n^2 - \alpha^2)v_{n-1} \qquad (\alpha \in \mathbb{C}) \\
\text{pot-mVL}: &\quad w_{n,x} = (w_{n+1} + 2\alpha w_n)(w_{n-1}^{-1} w_n + 2\alpha) \\
\mathrm{mVL}^2: &\quad v_{n,x} = (v_n - \alpha)v_{n+1}(v_n + \alpha) - (v_n + \alpha)v_{n-1}(v_n - \alpha) \\
\mathrm{VL}^2: &\quad u_{n,x} = u^{\mathrm{T}}_{n+1} u_n - u_n u^{\mathrm{T}}_{n-1}
\end{aligned}$$

Substitutions:

$$\begin{aligned}
\mathrm{VL}^1 \leftarrow \mathrm{mVL}^1: &\quad u_n = (v_{n+1} + \alpha)(v_n - \alpha) \qquad \text{discrete Miura map} \\
\mathrm{mVL}^1 \leftarrow \text{pot-mVL}: &\quad v_n = w_{n+1} w_n^{-1} + \alpha \\
\text{pot-mVL} \rightarrow \mathrm{mVL}^2: &\quad v_n = w_n^{-1} w_{n+1} + \alpha \\
\mathrm{mVL}^2 \rightarrow \mathrm{VL}^2: &\quad \begin{cases} u_n = (v_n + \alpha)(v_{n-1} + \alpha) & \text{for even } n \\ u_n = (v^{\mathrm{T}}_n - \alpha)(v^{\mathrm{T}}_{n-1} - \alpha) & \text{for odd } n \end{cases}
\end{aligned}$$

Remark: an (incomplete) analogy with KdV

There is a sequence of substitutions

$$\text{KdV} \xleftarrow[\text{Miura map}]{u=v^2\pm v_x+\alpha} \text{mKdV}^1 \xleftarrow{v=w_xw^{-1}} \text{pot-mKdV} \xrightarrow{v=w^{-1}w_x} \text{mKdV}^2$$

between

$$\begin{aligned}
\text{KdV}: &\quad u_t = u_{xxx} - 3uu_x - 3u_xu\\
\text{mKdV}^1: &\quad v_t = v_{xxx} - 3v^2v_x - 3v_xv^2 - 6\alpha v_x\\
\text{pot-mKdV}: &\quad w_t = w_{xxx} - 3w_{xx}w^{-1}w_x - 6\alpha w_x\\
\text{mKdV}^2: &\quad v_t = v_{xxx} + 3[v, v_{xx}] - 6vv_xv - 6\alpha v_x
\end{aligned}$$

These equations can be obtained from the corresponding lattice equations by continuous limit, but no continuous analog of VL^2 is known.

Symmetries: basic derivations

- $\partial_x = \partial_{t_1}$, the lattice itself
- ∂_{t_2}, the simplest higher symmetry

$$\begin{aligned} \mathrm{VL}^1: \quad u_{n,t_2} &= (u_{n+2}u_{n+1} + u_{n+1}^2 + u_{n+1}u_n)u_n \\ &\qquad - u_n(u_n u_{n-1} + u_{n-1}^2 + u_{n-1}u_{n-2}) \end{aligned}$$

$$\begin{aligned} \mathrm{VL}^2: \quad u_{n,t_2} &= (u^{\mathrm{T}}_{n+1}u_{n+2} + (u^{\mathrm{T}}_{n+1})^2 + u_n u^{\mathrm{T}}_{n+1})u_n \\ &\qquad - u_n(u^{\mathrm{T}}_{n-1}u_n + (u^{\mathrm{T}}_{n-1})^2 + u_{n-2}u^{\mathrm{T}}_{n-1}) \end{aligned}$$

- ∂_{τ_1}, the classical scaling symmetry

$$u_{n,\tau_1} = u_n$$

- ∂_{τ_2}, the master-symmetry (nonlocal for VL^1, local for VL^2)

$$\begin{aligned} \mathrm{VL}^1: \quad & u_{n,\tau_2} = \big(n + \tfrac{3}{2}\big)u_{n+1}u_n + u_n^2 - \big(n - \tfrac{3}{2}\big)u_n u_{n-1} + [s_n, u_n], \\ & s_n - s_{n-1} = u_n \end{aligned}$$

$$\mathrm{VL}^2: \quad u_{n,\tau_2} = \big(n + \tfrac{3}{2}\big)u^{\mathrm{T}}_{n+1}u_n + u_n^2 - \big(n - \tfrac{3}{2}\big)u_n u^{\mathrm{T}}_{n-1}$$

Remark: associated systems

Due to the lattice, any variable u_{n+k} is an expression of u_n, u_{n+1} and their x-derivatives. Thence, any symmetry is equivalent to some coupled PDE system. It is a non-Abelian generalization of the Levi system (Levi 1981, V.A. & Sokolov arXiv: 2008.09174). The map $n \to n+1$ defines a Bäcklund transformation for this system.

For VL^1, the pair $(p,q)=(u_n,u_{n+1})$ satisfies, for any n, the system

$$\begin{cases} q_{t_2} = \ q_{xx} + 2q_xq + 2(qp)_x + 2[qp,q], \\ p_{t_2} = -p_{xx} + 2pp_x + 2(qp)_x + 2[qp,p]. \end{cases}$$

For VL^2, the pair $(p,q)=(u_n,u^{\mathrm{T}}_{n+1})$ satisfies

$$\begin{cases} q_{t_2} = \ q_{xx} + 2q_xq + 2(pq)_x + 2[pq,q], \\ p_{t_2} = -p_{xx} + 2p_xp + 2(qp)_x + 2[p,qp]. \end{cases}$$

Symmetries and constraints

Like for KdV, there exists an infinite hierarchy of flows:

$$[\partial_{t_i}, \partial_{t_j}] = 0, \quad [\partial_{\tau_i}, \partial_{t_j}] = j\partial_{t_{j+i-1}},$$
$$[\partial_{\tau_i}, \partial_{\tau_j}] = (j-i)\partial_{\tau_{j+i-1}}, \qquad i, j \geq 1.$$

We only use symmetries that contain u_{n+k} with $|k| \leq 2$.

Any linear combination of derivations

$$\partial_t = \mu_1(x\partial_{t_2} + \partial_{\tau_2}) + \mu_2(x\partial_x + \partial_{\tau_1}) + \mu_3\partial_{t_2} + \mu_4\partial_x$$

commute with ∂_x. Therefore, the stationary equation

$$\partial_t(u_n) = 0$$

is a constraint consistent with the lattice.

Up to equivalence transformations, there are three different cases which lead to (non-Abelian) Painlevé equations:

$$\begin{array}{llll} & 2(x\partial_x+\partial_{\tau_1}) \quad +\partial_{t_2} & =0 & \to \quad \mathrm{dP}_1+\mathrm{P}_4\\ x\partial_{t_2}+\partial_{\tau_2} & +\mu(x\partial_x+\partial_{\tau_1}) & +\nu\partial_x=0 & \to \quad \mathrm{dP}_{34}+\mathrm{P}_5\\ x\partial_{t_2}+\partial_{\tau_2} & & +\nu\partial_x=0 & \to \quad \mathrm{dP}_{34}+\mathrm{P}_3 \end{array}$$

- In all cases, we start from some 5-point OΔE

$$f_n(u_{n-2},u_{n-1},u_n,u_{n+1},u_{n+2};x,\mu,\nu)=0.$$

- It admits a reduction of order due to *partial* first integrals (pfi).
- The final result is a discrete Painlevé equation

$$g_n(u_{n-1},u_n,u_{n+1};x,\mu,\nu,\varepsilon,\delta)=0.$$

- It defines a subclass of special solutions of the original equation. Additional constants $\varepsilon,\delta\in\mathbb{C}$ replace two matrix initial data.
- The x-dynamics is also consistent with pfi. The VL is reduced to an ODE system for (u_n,u_{n+1}) which is equivalent to a continuous Painlevé equation.

Scaling reduction: $\partial_{t_2} + 2(x\partial_x + \partial_{\tau_1}) = 0 \quad \to \quad \mathrm{dP}_1 + \mathrm{P}_4$

$$\mathrm{VL}^1: \ (u_{n+2}u_{n+1} + u_{n+1}^2 + u_{n+1}u_n)u_n - u_n(u_nu_{n-1} + u_{n-1}^2 + u_{n-1}u_{n-2}) \\ + 2x(u_{n+1}u_n - u_nu_{n-1}) + 2u_n = 0,$$

$$\mathrm{VL}^2: \ (u^{\mathrm{T}}_{n+1}u_{n+2} + (u^{\mathrm{T}}_{n+1})^2 + u_nu^{\mathrm{T}}_{n+1})u_n - u_n(u^{\mathrm{T}}_{n-1}u_n + (u^{\mathrm{T}}_{n-1})^2 + u_{n-2}u^{\mathrm{T}}_{n-1}) \\ + 2x(u^{\mathrm{T}}_{n+1}u_n - u_nu^{\mathrm{T}}_{n-1}) + 2u_n = 0.$$

This can be represented as $F_{n+1}u_n - u_nF_{n-1} = 0$.

The equality $F_n = 0$ is pfi. Its consistency with D_x is due to identities:

- $F_{n,x} = (F_{n+1} - F_n)u_n + u_n(F_n - F_{n-1})$ for VL^1
- $F_{n,x} = (F^{\mathrm{T}}_{n+1} + F_n)u_n - u_n(F_n + F^{\mathrm{T}}_{n-1})$ for VL^2

Two analogs of $\mathbf{dP}_1$

$$u_{n+1}u_n + u_n^2 + u_nu_{n-1} + 2xu_n + \gamma_n = 0, \qquad \mathrm{dP}_1^1$$

$$u^{\mathrm{T}}_{n+1}u_n + u_n^2 + u_nu^{\mathrm{T}}_{n-1} + 2xu_n + \gamma_n = 0, \qquad \mathrm{dP}_1^2$$

$$\gamma_n := n - \nu + (-1)^n\varepsilon.$$

Continuous dynamics is as follows.

- VL^1, $(p,y)=(u_{n-1},u_n)$:

$$p_x=2yp+p^2+2xp+\gamma_{n-1},\quad y_x=-y^2-2yp-2xy-\gamma_n,$$

- VL^2, $(p,y)=(u^{\mathrm{T}}_{n-1},u_n)$:

$$p_x=2py+p^2+2xp+\gamma_{n-1},\quad y_x=-y^2-2yp-2xy-\gamma_n.$$

Two analogs of $\mathbf{P}_4$

$$y''=\frac{1}{2}y'y^{-1}y'+[\kappa_i y-\gamma y^{-1},y']+\frac{3}{2}y^3+4xy^2+2(x^2-\alpha)y-2\gamma^2y^{-1},\quad \mathrm{P}_4^i$$

where $\alpha=\gamma_{n-1}-\gamma_n/2+1$, $\gamma=\gamma_n/2$,

$$\kappa_1=\frac{1}{2}\quad\text{and}\quad\kappa_2=-\frac{3}{2}.$$

- In the scalar case, this reduction was introduced by Its, Kitaev & Fokas (1990, 1991).
- Another non-Abelian version of dP_1 was studied by Cassatella-Contra, Mañas & Tempesta (2012, 2018):

$$u_{n+1} + u_n + u_{n-1} + 2x + \gamma_n u_n^{-1} = 0.$$

Master-symmetry reduction:

$$x\partial_{t_2} + \partial_{\tau_2} + \mu(x\partial_x + \partial_{\tau_1}) + \nu\partial_x = 0 \quad \to \quad \mathrm{dP}_{34} + \mathrm{P}_5 \text{ or } \mathrm{P}_3$$

The first step is easy (like in the previous case). It brings to 4-point equations

$$\begin{aligned}\mathrm{VL}^1:\ & x(u_{n+2}u_{n+1} + u_{n+1}^2 - u_n^2 - u_nu_{n-1}) - (2\mu x - n + \nu - \tfrac{3}{2})u_{n+1}\\ &\quad + (2\mu x - n + \nu + \tfrac{1}{2})u_n - \mu + 2(-1)^n\varepsilon = 0,\end{aligned}$$

$$\begin{aligned}\mathrm{VL}^2:\ & x\big(u^{\mathrm{T}}_{n+1}u_{n+2} + (u^{\mathrm{T}}_{n+1})^2 - u_n^2 - u_nu^{\mathrm{T}}_{n-1}\big) - (2\mu x - n + \nu - \tfrac{3}{2})u^{\mathrm{T}}_{n+1}\\ &\quad + (2\mu x - n + \nu + \tfrac{1}{2})u_n - \mu + 2(-1)^n\varepsilon = 0,\end{aligned}$$

where $\varepsilon \in \mathbb{C}$ is an integration constant. To obtain Painlevé equations, we need additional pfi.

In the scalar case, the above equation admits the integrating factor $xu_{n+1} + xu_n + n - \nu + \frac{1}{2}$ which brings to dP_{34}:

$$(z_{n+1} + z_n)(z_n + z_{n-1}) = 4x\frac{\mu z_n^2 + 2(-1)^n\varepsilon z_n + \delta}{z_n - n + \nu}, \quad z_n := 2xu_n + n - \nu.$$

Two analogs of dP_{34} for $\mu \neq 0$

$$(z_{n-1}+z_n)(z_n+(-1)^n\sigma+\omega)^{-1}(z_n+z_{n+1}) \\ = 4\mu x(z_n-n+\nu)^{-1}(z_n+(-1)^n\sigma-\omega), \qquad \mathrm{dP}^1_{34}$$

$$(z^{\mathrm{T}}_{n-1}+z_n)(z_n+(-1)^n(\sigma-\omega))^{-1}(z_n+z^{\mathrm{T}}_{n+1}) \\ = 4\mu x(z_n-n+\nu)^{-1}(z_n+(-1)^n(\sigma+\omega)) \qquad \mathrm{dP}^2_{34}$$

(where $\sigma=\varepsilon/\mu$, $\omega\in\mathbb{C}$).

Two analogs of dP_{34} for $\mu = 0$

$$\begin{cases}(z_{n+1}+z_n)(z_n-n+\nu)(z_n+z_{n-1}) = 4x(2\varepsilon z_n+\delta), & n=2k,\\ (z_n+z_{n-1})(z_{n+1}+z_n)(z_n-n+\nu) = 4x(-2\varepsilon z_n+\delta), & n=2k+1,\end{cases} \qquad \mathrm{d}\widetilde{\mathrm{P}}^1_{34}$$

$$(z^{\mathrm{T}}_{n+1}+z_n)(z_n-n+\nu)(z_n+z^{\mathrm{T}}_{n-1}) = 4x(2(-1)^n\varepsilon z_n+\delta). \qquad \mathrm{d}\widetilde{\mathrm{P}}^2_{34}$$

Equations dP^i_{34} and $\mathrm{d}\widetilde{\mathrm{P}}^i_{34}$ are consistent with VL^i. This gives rize to ODE systems for the variables $(q,p)=(z_n, z_n+z_{n+1})$ or $(z_n, z_n+z^{\mathrm{T}}_{n+1})$.

Two analogs of $\mathbf{P}_5$

$$\mathrm{dP}^1_{34} \to \quad \begin{cases} 2xq_x = p(q-n+\nu) - 4\mu x(q+\alpha)p^{-1}(q+\beta), \\ 2xp_x = pq+qp+p-p^2+4\mu x(p-2q-\alpha-\beta), \end{cases} \qquad \mathrm{P}^1_5$$

$$\mathrm{dP}^2_{34} \to \quad \begin{cases} 2xq_x = p(q-n+\nu) - 4\mu x(q+\alpha)p^{-1}(q+\beta), \\ 2xp_x = 2pq+p-p^2+4\mu x(p-2q-\alpha-\beta) \end{cases} \qquad \mathrm{P}^2_5$$

(in the scalar case, P_5 is satisfied by $y=1-4\mu xp^{-1}$).

Two analogs of $\mathbf{P}_3$

$$\mathrm{d}\widetilde{\mathrm{P}}^1_{34} \to \quad \begin{cases} 2xq_x = p(q-n+\nu) - 4xp^{-1}(2\varepsilon q+\delta), \\ 2xp_x = pq+qp+p-p^2-8\varepsilon x, \end{cases} \quad \text{(even } n\text{)} \qquad \mathrm{P}^1_3$$

$$\mathrm{d}\widetilde{\mathrm{P}}^2_{34} \to \quad \begin{cases} 2xq_x = p(q-n+\nu) - 4xp^{-1}(2(-1)^n\varepsilon q+\delta), \\ 2xp_x = 2pq+p-p^2-8(-1)^n\varepsilon x \end{cases} \qquad \mathrm{P}^2_3$$

(in the scalar case, P_3 is satisfied by $y=p/(2\xi)$, $x=\xi^2$).

Zero curvature representations

$$\mathrm{VL}^1: \quad u_{n,x} = u_{n+1}u_n - u_n u_{n-1} \quad \Leftrightarrow \quad L_{n,x} = U_{n+1}L_n - L_n U_n$$

$$L_n = \begin{pmatrix} \lambda & \lambda u_n \\ -1 & 0 \end{pmatrix}, \quad U_n = \begin{pmatrix} \lambda + u_n & \lambda u_n \\ -1 & u_{n-1} \end{pmatrix}$$

$$\mathrm{VL}^2: \quad u_{n,x} = u^{\mathrm{T}}_{n+1}u_n - u_n u^{\mathrm{T}}_{n-1} \quad \Leftrightarrow \quad L_{n,x} = U_{n+1}L_n + L_n U^{\mathrm{T}}_n$$

$$L_n = \begin{pmatrix} 1 & -\lambda \\ 0 & \lambda u_n \end{pmatrix}, \quad U_n = \begin{pmatrix} \frac{1}{2}\lambda & 1 \\ -\lambda u_{n-1} & -\frac{1}{2}\lambda - u_{n-1} + u^{\mathrm{T}}_n \end{pmatrix}$$

These are the compatiblity conditions, respectively, for

$$\Psi_{n+1} = L_n \Psi_n, \qquad \Psi_{n,x} = U_n \Psi_n$$

or for

$$\begin{aligned} \Psi_{2n+1} &= L_{2n}\Psi_{2n} & \Psi_{2n,x} &= -U^{\mathrm{T}}_{2n}\Psi_{2n}, \\ &= L^{\mathrm{T}}_{2n+1}\Psi_{2n+2}, & \Psi_{2n+1,x} &= U_{2n+1}\Psi_{2n+1}. \end{aligned}$$

Any derivation from $\mathrm{VL}^1/\mathrm{VL}^2$ hierarchy admits a representation

$$L_{n,t}+\kappa L_{n,\lambda}=V_{n+1}L_n-L_nV_n \qquad \text{or} \qquad L_{n,t}+\kappa L_{n,\lambda}=V_{n+1}L_n+L_nV_n^{\mathrm{T}},$$

with respective L_n and with certain V_n and $\kappa=\kappa(\lambda)$.

In both cases, we also have

$$U_{n,t}+\kappa U_{n,\lambda}=V_{n,x}+[V_n,U_n].$$

Therefore, for the stationary equation for ∂_t, we have the isomonodromic Lax pairs:

$$\kappa L_{n,\lambda}=V_{n+1}L_n-L_nV_n \qquad \text{or} \qquad \kappa L_{n,\lambda}=V_{n+1}L_n+L_nV_n^{\mathrm{T}}$$

for a discrete Painlevé equation and

$$\kappa U_{n,\lambda}=V_{n,x}+[V_n,U_n]$$

for a continuous one.

Explanation of dP^i_{34} partial first integral

Lemma. If $V_n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies Lax equations

$$V_{n,x} = [U_n, V_n], \qquad V_{n+1}L_n = L_nV_n \quad \text{or} \quad V_{n+1}L_n = -L_nV_n^{\mathrm{T}}$$

then its quasi-determinant $\Delta_n = b - ac^{-1}d$ is pfi.

Proof. It is easy to derive relations of the form

$$\Delta_{n,x} = f\Delta - \Delta g, \qquad \Delta_{n+1} = f\Delta_n g \quad \text{or} \quad \Delta_{n+1} = f\Delta_n^{\mathrm{T}} g$$

which imply that the constraint $\Delta = 0$ is preserved.

The constraint $x\partial_{t_2} + \partial_{\tau_2} + \mu(x\partial_x + \partial_{\tau_1}) + \nu\partial_x = 0$ admits the isomonodromic Lax pairs with $\kappa(\lambda) = \lambda^2 - 2\mu\lambda$. For $\lambda = 2\mu$, the matrix $V_n - \alpha I$ satisfies Lax equations and vanishing of its quasi-determinant gives exactly dP^i_{34}.